Exercise 11

Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A discovery revealed a parchment fragment that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.

Solution

Assume that the rate of mass decay is proportional to the amount of mass remaining at any given time.

$$\frac{dm}{dt} \propto -m$$

There's a minus sign here because mass is being lost as time increases. Change the proportionality to an equation by introducing a (positive) constant k.

$$\frac{dm}{dt} = -km$$

Divide both sides by m.

$$\frac{1}{m}\frac{dm}{dt} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dt}\ln m = -k$$

The function you have to differentiate to get $-k$ is $-kt + C$, where C is any constant.

$$\ln m = -kt + C$$

Exponentiate both sides.

$$e^{\ln m} = e^{-kt+C}$$

$$m(t) = e^C e^{-kt}$$

Use a new constant m_0 for e^C.

$$m(t) = m_0 e^{-kt} \tag{1}$$

The half-life is defined as the amount of time it takes for a sample to decay to half its mass, so set $m(5730) = m_0/2$ and solve the equation for k.

$$m(5730) = \frac{m_0}{2}$$

$$m_0 e^{-k(5730)} = \frac{m_0}{2}$$

$$e^{-5730k} = \frac{1}{2}$$

$$\ln e^{-5730k} = \ln \frac{1}{2}$$

$$(-5730k) \ln e = -\ln 2$$

$$k = \frac{\ln 2}{5730} \approx 0.000120968 \text{ year}^{-1}$$

As a result, equation (1) becomes

$$\begin{aligned} m(t) &= m_0 e^{-\left(\frac{\ln 2}{5730}\right)t} \\ &= m_0 e^{\ln 2^{-t/5730}} \\ &= m_0 (2)^{-t/5730}. \end{aligned}$$

To find how long it takes for the ^{14}C to reduce to 74% of its original amount, set $m(t) = 0.74 m_0$ and solve the equation for t.

$$m(t) = 0.74 m_0$$

$$m_0 (2)^{-t/5730} = 0.74 m_0$$

$$2^{-t/5730} = 0.74$$

$$\ln 2^{-t/5730} = \ln 0.74$$

$$\left(-\frac{t}{5730}\right) \ln 2 = \ln 0.74$$

$$t = -\frac{5730 \ln 0.74}{\ln 2} \approx 2489.13 \text{ years}$$

This is the age of the parchment.